Samelson products in p-regular exceptional Lie groups

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Abstract

The (non)triviality of Samelson products of the inclusions of the spheres into p-regular exceptional Lie groups is completely determined, where a connected Lie group is called p-regular if it has the p-local homotopy type of a product of spheres.

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1. Introduction and statement of the result

For a homotopy associative H-space with inverse X, the correspondence $X \wedge X \to X$, $(x, y) \mapsto xyx^{-1}y^{-1}$ induces a binary operation

$$\langle -, - \rangle : \pi_i(X) \otimes \pi_j(X) \to \pi_{i+j}(X)$$

called the Samelson product in X. We consider the basic Samelson products in p-regular Lie groups. Let G be a compact simply connected Lie group. By the Hopf theorem, G has the rational homotopy type of the product $S^{2n_1-1} \times \cdots \times S^{2n_\ell-1}$, where $n_1 \le \cdots \le n_\ell$. The sequence $n_1, \ldots, n_\ell$ is called the type of G and is denoted by $\mathtt{t}(G)$. We here list the types of exceptional Lie groups.

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G	$\mathtt{t}(G)$	G	$\mathtt{t}(G)$
G_2	$2, 6$	E_6	$2, 5, 6, 8, 9, 12$
F_4	$2, 6, 8, 12$	E_7	$2, 6, 8, 10, 12, 14, 18$
		E_8	$2, 8, 12, 14, 18, 20, 24, 30$

We say that G is p-regular if it has the p-local homotopy type of a product of spheres. By the classical result of Serre, it is known that G is p-regular if and only if $p \geq n_\ell$, in which case

$$G_{(p)} \simeq S^{2n_1-1}_{(p)} \times \cdots \times S^{2n_\ell-1}_{(p)}.$$

Suppose that G is p-regular, and let ϵ_{2n_i-1} be the composite

$$S^{2n_i-1} \xrightarrow{\text{incl}} S^{2n_1-1}_{(p)} \times \cdots \times S^{2n_\ell-1}_{(p)} \simeq G_{(p)}$$

where if there are more than one i in $\mathtt{t}(G)$, we distinguish the corresponding ϵ_{2i-1} but not write it explicitly. The Samelson products $\langle \epsilon_{2i-1}, \epsilon_{2j-1} \rangle$ are fundamental in studying the homotopy (non)commutativity of $G_{(p)}$ as in [KK] and its applications (See [KKTh, KKTs, Th], for example). So we would like to determine their (non)triviality. In [B], Bott computes the Samelson products in the classical groups $\mathrm{U}(n)$ and $\mathrm{Sp}(n)$. Then by combining with the information of the p-primary component of the homotopy groups of spheres [To], the (non)triviality of the Samelson products $\langle \epsilon_{2i-1}, \epsilon_{2j-1} \rangle$ is completely determined when $G = \mathrm{SU}(n), \mathrm{Sp}(n), \mathrm{Spin}(2n+1)$, where $\mathrm{Sp}(n)_{(p)} \simeq \mathrm{Spin}(2n+1)_{(p)}$ as loop spaces by [F] since p is odd. For example, when $G = \mathrm{SU}(n)$ and $p \geq n$, the type of G is given by $2, \ldots, n$ and

$$\langle \epsilon_{2i-1}, \epsilon_{2j-1} \rangle \neq 0 \quad \text{if and only if} \quad i + j > p.$$

So apart from $\mathrm{Spin}(2n)$, all we have to consider is the exceptional Lie groups. The (non)triviality of the Samelson products $\langle \epsilon_{2i-1}, \epsilon_{2j-1} \rangle$ is known only in a few cases, and the most general result so far is:

Theorem 1.1 (Hamanaka and Kono [HK]). *Let G be a p-regular exceptional Lie group. If $i, j \in \mathtt{t}(G)$ satisfy $i + j = p + 1$, then $\langle \epsilon_{2i-1}, \epsilon_{2j-1} \rangle$ is nontrivial.*

Remark 1.2. The Samelson products in G_2 are first computed in [O], and some more Samelson products in E_7 and E_8 are computed in [KK].

Based on this result, Kono posed the following conjecture (in a private communication).

Conjecture 1.3. Let G be a p-regular exceptional Lie group. For $i, j \in \mathtt{t}(G)$, there exists $k \in \mathtt{t}(G)$ satisfying $i + j = k + p - 1$ if and only if $\langle \epsilon_{2i-1}, \epsilon_{2j-1} \rangle$ is nontrivial.

Notice that the only if part of the conjecture follows immediately from the information of the p-primary component of the homotopy groups of spheres [To] (cf. [KK]). We will prove the if part and obtain:

Theorem 1.4. *Conjecture 1.3 is true.*

The paper is structured as follows. In §2, we reduce the nontriviality of the Samelson products $\langle \epsilon_{2i-1}, \epsilon_{2j-1} \rangle$ in the p-regular Lie group G to a certain condition of the Steenrod operation $\mathcal{P}^1$ on the mod p cohomology of the classifying space BG. Then for a p-regular exceptional Lie group G, we compute the mod p cohomology of BG as the ring of invariants of the Weyl group of G. With this description of the mod p cohomology of BG, we compute the action of $\mathcal{P}^1$ on it. In §3, we prove that the above condition on $\mathcal{P}^1$ is satisfied to complete the proof of Theorem 1.4.

2. Mod p cohomology of BG

2.1. Reduction

Let G be a compact simply connected Lie group. We first reduce Theorem 1.4 to the action of the Steenrod operation $\mathcal{P}^1$ on the mod p cohomology of the classifying space BG as in [HK, KK]. Recall that if the integral homology of G has no p-torsion, the mod p cohomology of the classifying space BG is given by

$$H^*(BG; \mathbb{Z}/p) = \mathbb{Z}/p[x_{2i} \,|\, i \in \mathtt{t}(G)], \quad |x_j| = j. \tag{1}$$

When there are more than one i in $\mathtt{t}(G)$, we distinguish corresponding x_{2i} but do not write it explicitly as in the case of ϵ_{2i-1} in the preceding section.

Lemma 2.1. *Suppose that G is p-regular. For $i, j \in \mathtt{t}(G)$, if there is $k \in \mathtt{t}(G)$ such that $\mathcal{P}^1 x_{2k}$ involves $\lambda x_{2i} x_{2j}$ with $\lambda \neq 0$, then $\langle \epsilon_{2i-1}, \epsilon_{2j-1} \rangle$ is nontrivial.*

Proof. Let $\bar{\epsilon}_{2i} : S^{2i} \to BG_{(p)}$ be the adjoint of ϵ_{2i-1} for $i \in \mathtt{t}(G)$, and so we may assume that $\bar{\epsilon}_{2i}^*(x_{2i}) = u_{2i}$ for a generator u_{2i} of $H^{2i}(S^{2i}; \mathbb{Z}/p)$. Assume that the Samelson product $\langle \epsilon_{2i-1}, \epsilon_{2j-1} \rangle$ is trivial, which is equivalent to the triviality of the Whitehead product $[\bar{\epsilon}_{2i}, \bar{\epsilon}_{2j}]$ by the adjointness of Samelson products and Whitehead products. Then the map $\bar{\epsilon}_{2i} \vee \bar{\epsilon}_{2j} : S^{2i} \vee S^{2j} \to BG_{(p)}$ extends to a map $\mu : S^{2i} \times S^{2j} \to BG_{(p)}$, up to homotopy. Hence since $\mathcal{P}^1 x_{2k}$ involves $\lambda x_{2i} x_{2j}$ with $\lambda \neq 0$, we have

$$\mu^*(\mathcal{P}^1 x_{2k}) = \mu^*(\lambda x_{2i} x_{2j}) = \lambda u_{2i} \times u_{2j} \neq 0.$$

On the other hand, by the naturality of $\mathcal{P}^1$, we also have

$$\mu^*(\mathcal{P}^1 x_{2k}) = \mathcal{P}^1 \mu^*(x_{2k}) = 0$$

since $\mathcal{P}^1$ is trivial on $H^*(S^{2i} \times S^{2j}; \mathbb{Z}/p)$, which is a contradiction. Therefore the proof is completed. □

By Lemma 2.1, we obtain the if part of Theorem 1.4 by the following.

Theorem 2.2. *Let G be a p-regular exceptional Lie group. If $i, j, k \in \mathtt{t}(G)$ satisfy $i + j = k + p - 1$, $\mathcal{P}^1 x_{2k}$ involves $\lambda x_{2i} x_{2j}$ with $\lambda \neq 0$.*

The rest of this paper is devoted to prove Theorem 2.2.

2.2. Generators

In this subsection, we choose generators of the mod p cohomology of BG. We set notation. Hereafter, let p be a prime greater than 5. Recall that the integral homology of G is p-torsion free for $p > 5$, and so the mod p cohomology of BG is given as (1). For a homomorphism $\rho : H \to K$ between Lie groups, we denote the induced map $BH \to BK$ ambiguously by ρ.

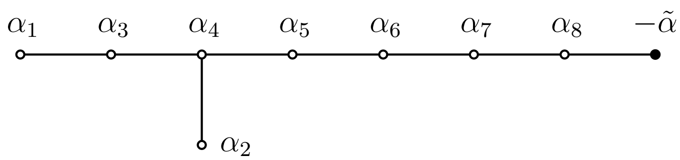
We first choose generators of the mod p cohomology of $B\mathrm{E}_8$. Let T be a maximal torus of E_8. Then as in [MT], since $p > 5$, the inclusion $T \to \mathrm{E}_8$ induces an isomorphism

$$H^*(B\mathrm{E}_8; \mathbb{Z}/p) \xrightarrow{\cong} H^*(BT; \mathbb{Z}/p)^{W(\mathrm{E}_8)}, \tag{2}$$

where the right hand side is the ring of invariants of the Weyl group $W(\mathrm{E}_8)$. We calculate invariants of $W(\mathrm{E}_8)$ through a maximal rank subgroup of E_8. Let $\epsilon_1, \ldots, \epsilon_8$ be the standard basis of $\mathbb{R}^8$ which is regarded as the Lie algebra of T. As in [MT], we choose simple roots of E_8 as

$$\alpha_1 = \frac{1}{2}(\epsilon_1 + \epsilon_8) - \frac{1}{2}(\epsilon_2 + \epsilon_3 + \epsilon_4 + \epsilon_5 + \epsilon_6 + \epsilon_7), \quad \alpha_2 = \epsilon_1 + \epsilon_2, \quad \alpha_i = \epsilon_{i-1} - \epsilon_{i-2} \quad (3 \leq i \leq 8),$$

by which the extended Dynkin diagram of E_8 is described as



where $\tilde{\alpha}$ is the dominant root. Removing α_1 from the diagram, we get the maximal rank subgroup of E_8 which is of type D_8. Then there is a homomorphism $\rho_1 : \mathrm{Spin}(16) \to \mathrm{E}_8$ which induces a monomorphism

$$\rho_1^* : H^*(B\mathrm{E}_8; \mathbb{Z}/p) \to H^*(B\mathrm{Spin}(16); \mathbb{Z}/p).$$

By putting $t_1 = -\epsilon_1$, $t_8 = -\epsilon_8$ and $t_i = \epsilon_i$ $(2 \le i \le 7)$, $H^*(BT; \mathbb{Z}/p)$ is identified with the polynomial ring $\mathbb{Z}/p[t_1, \dots, t_8]$. Let c_i and p_i be the i-th elementary symmetric functions in $t_1, \dots, t_8$ and in $t_1^2, \dots, t_8^2$, respectively. As in (2), we have an isomorphism

$$H^*(B\mathrm{Spin}(16); \mathbb{Z}/p) \xrightarrow{\cong} \mathbb{Z}[t_1, \dots, t_8]^{W(\mathrm{D}_8)} = \mathbb{Z}/p[p_1, \dots, p_7, c_8],$$

and then since $W(\mathrm{E}_8)$ is generated by $W(\mathrm{D}_8)$ and the reflection φ corresponding to the simple root α_1, it follows from (2) that

$$H^*(B\mathrm{E}_8; \mathbb{Z}/p) \cong \mathbb{Z}/p[p_1, \dots, p_7, c_8] \cap \mathbb{Z}/p[t_1, \dots, t_8]^{\varphi}. \tag{3}$$

Hence generators of $H^*(B\mathrm{E}_8; \mathbb{Z}/p)$ are chosen as elements of $\mathbb{Z}/p[p_1, \dots, p_7, c_8]$ which are invariant under φ. In [HK], the action of φ on $p_1, \dots, p_8, c_8 \in \mathbb{Z}/p[t_1, \dots, t_8]$ is described as

$$\varphi(p_1) = p_1, \quad \varphi(p_i) \equiv p_i + h_i c_1, \quad \varphi(c_8) \equiv c_8 - \frac{1}{4} c_7 c_1 \mod (c_1^2)$$

for $2 \le i \le 8$, where

$$h_2 = \frac{3}{2} c_3, \qquad h_3 = -\frac{1}{2}(5c_5 + c_3 c_2), \qquad h_4 = \frac{1}{2}(7c_7 + 3c_5 c_2 - c_4 c_3),$$
$$h_5 = -\frac{1}{2}(5c_7 c_2 - 3c_6 c_3 + c_5 c_4), \quad h_6 = -\frac{1}{2}(5c_8 c_3 - 3c_7 c_4 + c_6 c_5), \quad h_7 = \frac{1}{2}(3c_8 c_5 - c_7 c_6).$$

We put

$$
\begin{aligned}
\hat{x}_4 =& p_1,\\
\hat{x}_{16} =& 12p_4 - \frac{18}{5}p_3p_1 + p_2^2 + \frac{1}{10}p_2p_1^2 + 168c_8,\\
\hat{x}_{24} =& 60p_6 - 5p_5p_1 - 5p_4p_2 + 3p_3^2 - p_3p_2p_1 + \frac{5}{36}p_2^3 + 110c_8p_2,\\
\hat{x}_{28} =& 480p_7 + 40p_5p_2 - 12p_4p_3 - p_3p_2^2 - 3p_4p_2p_1 + \frac{24}{5}p_3^2p_1 + \frac{11}{36}p_2^3p_1 + 312c_8p_3 - 82c_8p_2p_1,\\
\hat{x}_{36} =& 480p_7p_2 + 72p_6p_3 - 30p_5p_4 - \frac{25}{2}p_5p_2^2 + 9p_4p_3p_2 - \frac{18}{5}p_3^3 - \frac{1}{4}p_3p_2^3 + 1020c_8p_5 + 102c_8p_3p_2\\
&- 42p_6p_2p_1 + 9p_5p_3p_1 - \frac{3}{2}p_4p_2^2p_1 + \frac{9}{5}p_3^2p_2p_1 + \frac{1}{24}p_2^4p_1 - 330c_8p_4p_1 - \frac{89}{2}c_8p_2^2p_1 - 300c_8^2p_1\\
&+ \frac{89}{4}p_5p_2p_1^2 - \frac{15}{2}p_4p_3p_1^2 - \frac{11}{20}p_3p_2^2p_1^2 + 156c_8p_3p_1^2 + \frac{5}{16}p_4p_2p_1^3 + \frac{9}{8}p_3^2p_1^3 + \frac{27}{320}p_2^3p_1^3\\
&- \frac{323}{8}c_8p_2p_1^3 - \frac{195}{32}p_5p_1^4 - \frac{13}{64}p_3p_2p_1^4 - \frac{7}{192}p_2^2p_1^5 + \frac{195}{32}c_8p_1^5 + \frac{3}{32}p_3p_1^6 - \frac{1}{1024}p_2p_1^7,\\
\hat{x}_{40} =& 480p_7p_3 + 50p_6p_2^2 + 50p_5^2 - 10p_5p_3p_2 - \frac{25}{2}p_4^2p_2 + 9p_4p_3^2 - \frac{25}{36}p_4p_2^3 + \frac{3}{4}p_3^2p_2^2 + \frac{25}{864}p_2^5\\
&+ 2400c_8p_6 + 250c_8p_4p_2 + 3550c_8^2p_2 + 6c_8p_3^2 - \frac{175}{18}c_8p_2^3,\\
\hat{x}_{48} =& -200p_7p_5 - 60p_7p_3p_2 + 3p_6p_3^2 + \frac{25}{9}p_6p_2^3 + \frac{25}{3}p_5^2p_2 - \frac{5}{2}p_5p_4p_3 - \frac{25}{24}p_5p_3p_2^2 - \frac{25}{48}p_4^2p_2^2\\
&+ p_4p_3^2p_2 + \frac{25}{864}p_4p_2^4 - \frac{3}{10}p_3^4 - \frac{1}{36}p_3^2p_2^3 - \frac{25}{62208}p_2^6 - 400c_8p_6p_2 - 115c_8p_5p_3 - \frac{25}{12}c_8p_4p_2^2\\
&+ 3c_8p_3^2p_2 + \frac{25}{27}c_8p_2^4 + 75c_8p_4^2 - 300c_8^2p_4 - \frac{1525}{12}c_8^2p_2^2 + 300c_8^3.
\end{aligned}
$$

We shall prove that the elements $\hat{x}_i$ are invariant under φ and algebraically independent, implying that they are generators of $H^*(B\mathrm{E}_8;\mathbb{Z}/p)$ through the isomorphism (3). Hamanaka and Kono [HK] calculate φ-invariants in dimension 4, 16 and 24 as follows.

Proposition 2.3 (Hamanaka and Kono [HK])**.** *Let $\bar{x}_i \in \mathbb{Z}/p[p_1,\dots,p_7,c_8]$ with $|\bar{x}_i| = i$.*

1. *If $\varphi(\bar{x}_i) \equiv \bar{x}_i \mod (c_1^2)$ in $\mathbb{Z}/p[t_1,\dots,t_8]$ for $i = 4, 16$, then*

$$\bar{x}_4 = \alpha\hat{x}_4 \quad \textit{and} \quad \bar{x}_{16} = \beta\hat{x}_{16} + \gamma\hat{x}_4^4 \quad (\alpha,\beta,\gamma \in \mathbb{Z}/p).$$

2. *If* $\varphi(\bar{x}_{24}) \equiv \bar{x}_{24} \mod (c_1^2, c_2^2)$ *in* $\mathbb{Z}/p[t_1, \ldots, t_8]$*, then*

$$\bar{x}_{24} \equiv \alpha \hat{x}_{24} \quad (\alpha \in \mathbb{Z}/p).$$

We further calculate φ-invariants in dimension $28, 36, 40, 48$, where a partial calculation in dimension 28 is given in [KK].

Proposition 2.4 (cf. [KK])**.** *Let* $\bar{x}_i \in \mathbb{Z}/p[p_1, \ldots, p_7, c_8]$ *with* $|\bar{x}_i| = i$*.*

1. *If* $\varphi(\bar{x}_{28}) \equiv \bar{x}_{28} \mod (c_1^2, c_2^2)$ *in* $\mathbb{Z}/p[t_1, \ldots, t_8]$*, then*

$$\bar{x}_{28} \equiv \alpha \hat{x}_{28} + \beta \hat{x}_4 \hat{x}_{24} \mod (p_1^2) \quad (\alpha, \beta \in \mathbb{Z}/p).$$

2. *If* $\varphi(\bar{x}_{36}) \equiv \bar{x}_{36} \mod (c_1^2)$ *in* $\mathbb{Z}/p[t_1, \ldots, t_8]$*, then*

$$\bar{x}_{36} = \alpha_1 \hat{x}_{36} + \alpha_2 \hat{x}_4 \hat{x}_{16}^2 + \alpha_3 \hat{x}_4^2 \hat{x}_{28} + \alpha_4 \hat{x}_4^3 \hat{x}_{24} + \alpha_5 \hat{x}_4^5 \hat{x}_{16} + \alpha_6 \hat{x}_4^9 \quad (\alpha_i \in \mathbb{Z}/p).$$

3. *If* $\varphi(\bar{x}_i) \equiv \bar{x}_i \mod (c_1^2, c_2)$ *in* $\mathbb{Z}/p[t_1, \ldots, t_8]$ *for* $i = 40, 48$*, then*

$$\bar{x}_{40} \equiv \alpha_1 \hat{x}_{40} + \alpha_2 \hat{x}_{24} \hat{x}_{16}, \quad \bar{x}_{48} \equiv \beta_1 \hat{x}_{48} + \beta_2 \hat{x}_{24}^2 + \beta_3 \hat{x}_{16}^3 \mod (p_1) \quad (\alpha_i, \beta_i \in \mathbb{Z}/p).$$

Proof. The proof is the same as Proposition 2.3 given in [HK], and we only consider $\bar{x}_{28}$ since other cases are analogous. Excluding the indeterminacy $\hat{x}_4 \hat{x}_{24}$, we may suppose that $\bar{x}_{28}$ is a linear combination

$$\lambda_1 p_7 + \lambda_2 p_5 p_2 + \lambda_3 p_4 p_3 + \lambda_4 p_4 p_2 p_1 + \lambda_5 p_3^2 p_1 + \lambda_6 p_3 p_2^2 + \lambda_7 p_2^3 p_1 + \lambda_8 c_8 p_3 + \lambda_9 c_8 p_2 p_1$$

for $\lambda_i \in \mathbb{Z}/p$. By the congruence $\varphi(\bar{x}_{28}) \equiv \bar{x}_{28} \mod (c_1^2, c_2^2)$ and the equality $p_i = \sum_{j+k=2i} (-1)^{i+j} c_j c_k$, we get linear equations in $\lambda_1, \ldots, \lambda_9$. Solving these equations, we see that $\bar{x}_{28} \equiv \alpha \hat{x}_{28} \mod (c_1^2, c_2^2)$, thus the proof is completed since the intersection of the ideal (c_1^2, c_2^2) and the subring $\mathbb{Z}/p[p_1, \ldots, p_7, c_8]$ of $\mathbb{Z}/p[t_1, \ldots, t_8]$ is the ideal (p_1^2) in $\mathbb{Z}/p[p_1, \ldots, p_7, c_8]$. $\square$

As an immediate consequence of Proposition 2.3 and 2.4, we obtain:

Corollary 2.5. *We can choose a generator* x_i *of* $H^*(B\mathrm{E}_8; \mathbb{Z}/p)$ *for* $i \neq 60$ *in such a way that*

$$\rho_1^*(x_i) = \hat{x}_i \quad (i = 4, 16, 36), \quad \rho_1^*(x_i) \equiv \hat{x}_i \mod (p_1^2) \quad (i = 24, 28)$$
$$\rho_1^*(x_i) \equiv \hat{x}_i \mod (p_1) \quad (i = 40, 48).$$

Hereafter, we choose generators of $H^*(B\mathrm{E}_8, \mathbb{Z}/p)$ as in Corollary 2.5. From these generators, we next choose generators of $H^*(BG; \mathbb{Z}/p)$ for $G = \mathrm{F}_4, \mathrm{E}_6, \mathrm{E}_7$. Recall that there is a commutative diagram of canonical homomorphisms

$$\begin{tikzcd}
\mathrm{F}_4 \arrow[r, "\alpha_3"] & \mathrm{E}_6 \arrow[r, "\alpha_2"] & \mathrm{E}_7 \arrow[r, "\alpha_1"] & \mathrm{E}_8 \\
\mathrm{Spin}(9) \arrow[u, "\rho_4"] \arrow[r, "\theta_3"] & \mathrm{Spin}(10) \arrow[u, "\rho_3"] \arrow[r, "\theta_2"] & \mathrm{Spin}(12) \arrow[u, "\rho_2"] \arrow[r, "\theta_1"] & \mathrm{Spin}(16). \arrow[u, "\rho_1"]
\end{tikzcd} \tag{4}$$

Let us consider the induced map of arrows in the mod p cohomology of the classifying spaces. Obviously, we have

$$\theta_1^*(p_i) = p_i \ (i = 1, 2, 3, 4, 5), \quad \theta_1^*(p_6) = c_6^2, \quad \theta_1^*(p_7) = 0, \quad \theta_1^*(c_8) = 0, \tag{5}$$
$$\theta_2^*(p_i) = p_i \ (i = 1, 2, 3, 4), \quad \theta_2^*(p_5) = c_5^2, \quad \theta_2^*(c_6) = 0, \tag{6}$$
$$\theta_3^*(p_i) = p_i \ (i = 1, 2, 3, 4), \quad \theta_3^*(c_5) = 0. \tag{7}$$

To determine the induced map of α_i, we recall the results of [A, C, N, TW, W].

Proposition 2.6. 1. *$H^*(\mathrm{E}_6/\mathrm{Spin}(10); \mathbb{Z}/p) = \mathbb{Z}/p[y_8]/(y_8^3) \otimes \Lambda(y_{17})$, $|y_i| = i$.*

2. *$H^*(\mathrm{E}_6/\mathrm{F}_4; \mathbb{Z}/p) = \Lambda(z_9, z_{17})$, $|z_i| = i$.*
3. *$\widetilde{H}^*(\mathrm{E}_7/\mathrm{E}_6; \mathbb{Z}/p) = \mathbb{Z}/p\langle z_{10}, z_{18}\rangle$, $|z_i| = i$ for $* < 37$.*
4. *$H^*(\mathrm{E}_8/\mathrm{E}_7; \mathbb{Z}/p) = \mathbb{Z}/p[z_{12}, z_{20}]$, $|z_i| = i$ for $* < 40$, .*

We next choose generators of $H^*(BG; \mathbb{Z}/p)$ for $G \neq \mathrm{E}_8$. Let

$$\hat{x}_{10} = c_5, \quad \hat{x}_{12} = -6p_3 + p_2 p_1 - 60c_6, \quad \hat{x}_{18} = p_2 c_5 \quad \text{and} \quad \hat{x}_{20} = p_5 + p_2 c_6.$$

We abbreviate $\theta_i(\hat{x}_j)$ by $\hat{x}_j$.

Corollary 2.7. *We can choose a generator x_i of $H^*(B\mathrm{E}_7; \mathbb{Z}/p)$ so that*

$$\rho_2^*(x_i) = \hat{x}_i \quad (i = 4, 12, 16, 36) \quad \textit{and} \quad \rho_2^*(x_i) \equiv \hat{x}_i \mod (p_1^2) \quad (i = 20, 24, 28).$$

Proof. Consider the Serre spectral sequence of the homotopy fiber sequence $\mathrm{E}_8/\mathrm{E}_7 \to B\mathrm{E}_7 \to B\mathrm{E}_8$. Then by Proposition 2.6, we get $\alpha_1^*(x_i) = x_i$ for $i = 4, 16, 24, 28, 36$, hence the desired result for $\rho_2^*(x_i)$ by Corollary 2.5. As in [BH], we can choose a generator x_{12} of $H^*(B\mathrm{F}_4; \mathbb{Z}/p)$ so that $\rho_4^*(x_{12}) = -6p_3 + p_2 p_1$. On the other hand, it is calculated in [N] that $\rho_2^*(x_{12}) \equiv -6p_3 - 60c_6$

modulo decomposables. Then we get $\rho_2^*(x_{12}) = \hat{x}_{12}$ by (6) and (7). By the Serre spectral sequence of the homotopy fiber sequence $\mathrm{E}_6/\mathrm{Spin}(10) \to B\mathrm{Spin}(10) \to B\mathrm{E}_6$ and Proposition 2.6, we have $\rho_3^*(x_{10}) \neq 0$. Then for a degree reason, we may choose $x_{10} \in H^*(B\mathrm{E}_6; \mathbb{Z}/p)$ so that $\rho_3^*(x_{10}) = c_5$. Consider next the Serre spectral sequence of the homotopy fiber sequence $\mathrm{E}_7/\mathrm{E}_6 \to B\mathrm{E}_6 \to B\mathrm{E}_7$. Then it follows from Proposition 2.6 that we may choose $x_{20} \in H^*(B\mathrm{E}_7; \mathbb{Z}/p)$ so that $\alpha_2^*(x_{20}) = x_{10}^2$, hence $\rho_2^*(x_{20}) \equiv p_5 + \alpha p_2 c_6 \mod (p_1^2)$ by (6), where $\alpha \in \mathbb{Z}/p$. For a degree reason, we have $\alpha_1^*(x_{40}) \equiv \lambda x_{20}^2 \mod (x_4, x_{12}, x_{16})$, hence

$$\theta_2^*(\hat{x}_{40}) = \lambda(p_5 + \alpha p_2 c_6)^2 \mod (\hat{x}_4, \hat{x}_{12}, \hat{x}_{16}).$$

Since $\theta_2^*(\hat{x}_{40}) \equiv 50p_5^2 - 10p_5p_3p_2 + \frac{1}{2}p_3^2p_2^2$ and $\hat{x}_{20}^2 \equiv p_5^2 - \frac{\alpha}{5}p_5p_3p_2 + \frac{\alpha^2}{100}p_3^2p_2^2 \mod (\hat{x}_4, \hat{x}_{12}, \hat{x}_{16})$, we get $\alpha = 1$ and $\lambda = 50$. □

Corollary 2.8. *We can choose a generator x_i of $H^*(B\mathrm{E}_6; \mathbb{Z}/p)$ so that*

$$\rho_3^*(x_i) = \hat{x}_i \quad (i = 4, 10, 12, 16, 18) \quad \textit{and} \quad \rho_3^*(x_{24}) = \hat{x}_{24} \mod (p_1^2).$$

Proof. By the Serre spectral sequence of the homotopy fiber sequence $\mathrm{E}_7/\mathrm{E}_6 \to B\mathrm{E}_6 \to B\mathrm{E}_7$ together with Proposition 2.6 and Corollary 2.7, we get $\alpha_2^*(x_i) = x_i$ for $i = 4, 12, 16, 24$. Then we obtain the desired result for x_i ($i = 4, 12, 16, 24$) by Corollary 2.7. By Proposition 2.6, we have $\rho_3^*(x_{10}) \neq 0$, so we may put $\rho_3^*(x_{10}) = c_5$ for a degree reason. By Proposition 2.4, Corollary 2.7 and $\alpha_2 \circ \rho_3 = \rho_2 \circ \theta_2$, we see that $\rho_3^* \circ \alpha_2^*(x_{28})$ includes the term $p_2c_5^2$ which does not belong to $\rho_3^*(\mathbb{Z}/p[x_4, \ldots, \widehat{x_{18}}, \ldots, x_{24}])$. Then we get $\rho_3^*(x_{18}) \neq 0$, implying that we may put $\rho_3^*(x_{18}) = p_2c_5$ for a degree reason. □

Corollary 2.9. *We can choose a generator x_i of $H^*(B\mathrm{F}_4; \mathbb{Z}/p)$ so that*

$$\rho_4^*(x_i) = \hat{x}_i \quad (i = 4, 12, 16) \quad \textit{and} \quad \rho_4^*(x_{24}) \equiv \hat{x}_{24} \mod (p_1^2).$$

Proof. The result follows from the Serre spectral sequence of the homotopy fiber sequence $\mathrm{E}_6/\mathrm{F}_4 \to B\mathrm{F}_4 \to B\mathrm{E}_6$ together with Proposition 2.6 and Corollary 2.8. □

Recall that G_2 is a subgroup of $\mathrm{Spin}(7)$. We denote the inclusion $\mathrm{G}_2 \to \mathrm{Spin}(7)$ by ρ.

Proposition 2.10. *The induced map of $\rho : B\mathrm{G}_2 \to B\mathrm{Spin}(7)$ in mod p cohomology satisfies*

$$\rho^*(p_1) = x_4, \quad \rho^*(p_2) = 0 \quad \textit{and} \quad \rho^*(p_3) = x_{12}.$$

Proof. It is well known that $\mathrm{Spin}(7)/\mathrm{G}_2 = S^7$. Then by considering the Serre spectral sequence of the homotopy fiber sequence $\mathrm{Spin}(7)/\mathrm{G}_2 \to B\mathrm{G}_2 \to B\mathrm{Spin}(7)$, we obtain the desired result. $\square$

For the rest of this paper, we choose generators of $H^*(BG;\mathbb{Z}/p)$ as in Corollary 2.7, 2.8, 2.9, 2.10.

2.3. Calculation of $\mathcal{P}^1\rho_i^*(x_j)$

We first calculate the action of $\mathcal{P}^1$ on $H^*(B\mathrm{Spin}(2m);\mathbb{Z}/p)$. Recall that $H^*(B\mathrm{Spin}(2m);\mathbb{Z}/p) = \mathbb{Z}/p[p_1,\dots,p_{m-1},c_m]$ as above.

Lemma 2.11. *In $H^*(B\mathrm{Spin}(2m);\mathbb{Z}/p)$, we have*

$$\mathcal{P}^1 p_i = \sum_{i_1+2i_2+\cdots+mi_m=i+\frac{p-1}{2}} (-1)^{i_1+\cdots+i_m+\frac{p+1}{2}} \frac{(i_1+\cdots+i_m-1)!}{i_1!\cdots i_m!} \times \left(2i-1-\frac{\sum_{j=1}^{i-1}(2i+p-1-2j)i_j}{i_1+\cdots+i_m-1}\right) p_1^{i_1}\cdots p_m^{i_m}$$

and $\mathcal{P}^1 c_m = s_{p-1}c_m$, where $p_m = c_m^2$ and $s_k = t_1^k+\cdots+t_m^k$.

Proof. By [S], we have the mod p Wu formula

$$\mathcal{P}^1 c_i = \sum_{i_1+2i_2+\cdots+2mi_{2m}=i+p-1} (-1)^{i_1+\cdots+i_{2m}-1} \frac{(i_1+\cdots+i_{2m}-1)!}{i_1!\cdots i_{2m}!} \times \left(i-1-\frac{\sum_{j=2}^{i-1}(i+p-1-j)i_j}{i_1+\cdots+i_{2m}-1}\right) c_1^{i_1}\cdots c_{2m}^{i_{2m}}$$

in $H^*(B\mathrm{U}(2m);\mathbb{Z}/p)$. Since the natural map $\mathbf{c}\colon B\mathrm{Spin}(2m) \to B\mathrm{U}(2m)$ satisfies $\mathbf{c}^*(c_{2i}) = (-1)^i p_i$ and $\mathbf{c}^*(c_{2i+1}) = 0$, we obtain the first equation. The second equation is obvious. $\square$

We now calculate $\mathcal{P}^1\rho_i^*(x_j)$.

Proposition 2.12. *Define ideals I_j of $\mathbb{Z}/p[p_1,\dots,p_7,c_8]$ for $j=0,\dots,8$ as*

$I_0 = (p_1,p_2^2,p_3^3,p_4^2,p_6^2,c_8)$, $I_1 = I_0+(p_3,p_6)$, $I_2 = I_0+(p_2,p_3^2,p_4,p_7^2)$,
$I_3 = I_0+(p_2,p_3^2,p_6)$, $I_4 = I_0+(p_2,p_3^2,p_4)$, $I_5 = I_0+(p_2,p_3,p_4,p_6,p_7)$,
$I_6 = I_0+(p_2,p_3^2,p_4,p_6)$, $I_7 = I_0+(p_2,p_3^2,p_4,p_6,p_7^2)$, $I_8 = I_0+(p_2,p_4,p_7^4,\hat{x}_{24})$.

Then for a generator $x_k \in H^*(BE_8;\mathbb{Z}/p)$, we have the following table.

p	k	$\mathcal{P}^1\rho_1^*(x_k) \mod I$	I	p	k	$\mathcal{P}^1\rho_1^*(x_k) \mod I$	I
31	16	$9p_7^2p_5+24p_7p_5^2p_2+22p_5^3p_4$	I_1	37	4	$p_7^2p_5+34p_7p_5^2p_2+36p_5^3p_4$	I_1
	24	$28p_7p_6p_5p_3+16p_6p_5^3$	I_2		16	$8p_7^2p_5p_3+27p_7p_5^3+2p_5^3p_4p_3$	I_3
	28	$27p_7^2p_5p_3+30p_7p_5^3+30p_5^3p_4p_3$	I_3		24	$5p_7^3p_3+27p_7^2p_5^2+36p_6p_5^3p_3$	I_4
	36	$p_7^3p_3+10p_7^2p_5^2+6p_6p_5^3p_3$	I_4		28	$7p_5^5$	I_5
	40	$8p_5^5$	I_5		36	$20p_7^2p_5^2p_3+35p_7p_5^4$	I_6
	48	$4p_7^2p_5^2p_3+5p_7p_5^4$	I_6		48	$36p_7p_5^4p_3+3p_5^6$	I_7
41	4	$35p_7p_6p_5p_3+40p_6p_5^3$	I_2	43	4	$3p_7^2p_5p_3+p_7p_5^3+39p_5^3p_4p_3$	I_3
	16	$9p_7^3p_3+38p_7^2p_5^2+16p_6p_5^3p_3$	I_4		16	$9p_5^5$	I_5
	28	$7p_7^2p_5^2p_3+6p_7p_5^4$	I_6		24	$11p_7^2p_5^2p_3+40p_7p_5^4$	I_6
	40	$34p_7p_5^4p_3+16p_5^6$	I_7		36	$35p_7p_5^4p_3+42p_5^6$	I_7
47	4	$p_7^3p_3+25p_7^2p_5^2+43p_6p_5^3p_3$	I_4	53	4	$6p_7^2p_5^2p_3+p_7p_5^4$	I_6
	16	$35p_7^2p_5^2p_3+10p_7p_5^4$	I_6		16	$23p_7p_5^4p_3+39p_5^6$	I_7
	28	$17p_7p_5^4p_3+23p_5^6$	I_7	59	4	$5p_7p_5^4p_3+10p_5^6$	I_7

For $p=31$, we also have

$$\mathcal{P}^1\rho_1^*(x_{48}) \equiv 17p_7^3p_3^2+4p_7^2p_5^2p_3+5p_7p_5^4, \quad \mathcal{P}^2\rho_1^*(x_{48}) \equiv 26p_7^3p_5^3p_3^2+5p_7^2p_5^5p_3+8p_7p_5^7 \mod I_8.$$

Proof. For $i=4,16,24,28,36$, we have $\rho_1^*(x_i)\equiv\hat{x}_i \mod (p_1^2)$. Since $\mathcal{P}^1(p_1^2)\subset(p_1)$ by the Cartan formula, we have $\mathcal{P}^1\rho_1^*(x_i)\equiv\mathcal{P}^1\hat{x}_i \mod (p_1)$. For $i=40,48$, we analogously have $\mathcal{P}^1\rho_1^*(x_i)=\mathcal{P}^1\hat{x}_i+(\mathcal{P}^1p_1)q$ for some polynomial q in $p_2,\ldots,p_7,c_8$. For a degree reason, we have $q\equiv 0 \mod (p_1,p_2,p_3^2,p_4,p_6,c_8)$, implying that $\mathcal{P}^1\rho_1^*(x_i)\equiv\mathcal{P}^1\hat{x}_i \mod I$ for the prescribed ideal I. Thus in order to fill the table, we only need to calculate $\mathcal{P}^1\hat{x}_i$ by Lemma 2.11.

For $p=31$, we have $\mathcal{P}^1\rho_1^*(x_{48})\equiv\mathcal{P}^1\hat{x}_{48}+(\mathcal{P}^1p_1)q \mod (p_1)$ for some polynomial q in $p_2,\ldots,p_7,c_8$ as above. Since $\hat{x}_i\in I_8$ for $i=4,16,24,36$, we have $\mathcal{P}^1p_1\equiv 0 \mod I_8$ for a degree reason, hence $\mathcal{P}_1\rho_1^*(x_{48})\equiv\mathcal{P}^1\hat{x}_{48} \mod I_8$. Then we can calculate $\mathcal{P}^1\rho_1^*(x_{48}) \mod I_8$ by Lemma 2.11. Since $\mathcal{P}^2p_1=p_1^p$ and $\rho_1^*(x_{48})\equiv\hat{x}_{48} \mod (p_1)$, we have $\mathcal{P}^2\rho_1^*(x_{48})\equiv\mathcal{P}^2\hat{x}_{48} \mod (p_1)$. Now $\mathcal{P}^2\rho_1(x_{48})$ for $p=31$ can be calculated from Lemma 2.11 and the Adem relation $\mathcal{P}^1\mathcal{P}^1=2\mathcal{P}^2$. □

Quite similarly to Proposition 2.12, we can calculate $\mathcal{P}^1\rho_i^*(x_j)$ for $G=E_7,E_6$.

Proposition 2.13. *For a generator $x_k \in H^*(B\mathrm{E}_7; \mathbb{Z}/p)$, we have the following table.*

p	k	$\mathcal{P}^1\rho_2^*(x_k) \mod I$	I
19	12	$18p_5^2p_2 + 3p_5p_4p_3 + 15p_5p_3p_2^2 + 10p_4^3 + 17p_4^2p_2^2 + 6p_4p_2^4 + 15p_2^6$	(p_1, p_3^2, c_6)
	16	$11p_5p_4^2 + 16p_5p_4p_2^2 + 15p_5p_2^4$	(p_1, p_3, c_6)
	20	$p_5^2p_4 + 18p_5^2p_2^2 + 17p_5p_4p_3p_2 + p_5p_3p_2^3 + 4c_6p_5p_4p_2 + 12c_6p_5p_2^3$ $+16c_6p_4^2p_3 + 8p_4c_6p_3p_2^2 + 7c_6p_3p_2^4$	(p_1, p_3^2, c_6^2)
	24	$13p_5p_4^2p_2 + 7p_5p_4p_2^3 + 8p_5p_2^5$	(p_1, p_3, p_5^2, c_6)
	28	$14p_5^2p_4p_2 + p_5^2p_2^3 + 8p_5p_4^2p_3 + 10p_5p_4p_3p_2^2 + 17p_5p_3p_2^4 + p_4^4 + 9p_4^3p_2^2$ $+6p_4^2p_2^4 + p_4p_2^6 + 3p_2^8$	(p_1, p_3^2, c_6^2)
	36	$9p_5^2p_4^2 + 4p_5^2p_4p_2^2 + 6p_5^2p_2^4 + 17p_5p_4^2p_3p_2 + 15p_5p_3p_2^5 + 4p_4^4p_2 + 5p_4^3p_2^3$ $+2p_4^2p_2^5 + 11p_4p_2^7 + 3p_2^9$	(p_1, p_3^2, c_6^2)
23	4	$22p_5^2p_2 + 21p_5p_4p_3 + 3p_5p_3p_2^2 + 15p_4^3 + 13p_4^2p_2^2 + 22p_4p_2^4 + 4p_2^6$	(p_1, p_3^2, c_6)
	12	$7p_5^2p_4 + 6p_5^2p_2^2 + 14p_5p_4p_3p_2 + 13p_5p_3p_2^3 + 10p_4^3p_2 + 18p_4^2p_2^3 + 21p_4p_2^5$ $+4p_2^7 + 14c_6p_5p_4p_2 + 16c_6p_5p_2^3 + 7c_6p_4^2p_3 + 2p_4c_6p_3p_2^2 + 7c_6p_3p_2^4$	(p_1, p_3^2, c_6^2)
	16	$3p_5p_4^2p_2 + 20p_5p_4p_2^3 + 19p_5p_2^5$	(p_1, p_3, p_5^2, c_6)
	28	$9p_5^2p_4^2 + 3p_5^2p_4p_2^2 + 2p_5^2p_2^4 + 10p_5p_4^2p_3p_2 + 10p_5p_4p_3p_2^3 + 8p_5p_3p_2^5$ $+14p_4^4p_2 + 15p_4^3p_2^3 + 14p_2^9 + 9p_4^2p_2^5 + 15p_4p_2^7$	(p_1, p_3^2, c_6^2)
29	4	$26p_5p_4^2p_2 + 4p_5p_4p_2^3 + 28p_5p_2^5$	(p_1, p_3, p_5^2, c_6)
	16	$19p_5^2p_4^2 + p_5^2p_4p_2^2 + 19p_5^2p_2^4 + 10p_5p_4^2p_3p_2 + 6p_5p_4p_3p_2^3 + 13p_5p_3p_2^5$ $+p_4^4p_2 + 7p_4^3p_2^3 + 2p_4^2p_2^5 + 16p_4p_2^7 + 21p_2^9$	(p_1, p_3^2, c_6^2)
31	12	$p_5^3p_3 + 17p_5^2p_4^2 + 10p_5^2p_4p_2^2 + 28p_5^2p_2^4 + 4p_5p_4^2p_3p_2 + 18p_5p_4p_3p_2^3$ $+21p_2p_4^4 + 3p_4^3p_2^3 + 6p_4p_2^7 + 4p_5^3p_2^9 + 10c_6p_5^3 + 3c_6p_5^2p_3p_2 + 3c_6p_5p_4^2p_2$ $+27c_6p_5p_4p_2^3 + c_6p_5p_2^5 + c_6p_4^3p_3 + 25c_6p_4^2p_3p_2^2 + 5c_6p_4p_3p_2^4 + 30c_6p_3p_2^6$	(p_1, p_3^2, c_6^2)

Proposition 2.14. *For a generator $x_k \in H^*(B\mathrm{E}_6; \mathbb{Z}/p)$, we have the following table.*

p	k	$\mathcal{P}^1\rho_3^*(x_k) \mod I$	I
13	10	$6c_5p_4p_2 + 11c_5p_2^3$	(p_1, p_3^2, c_5^2)
	12	$10p_4p_3p_2 + 12p_3p_2^3 + 4c_5^2p_4 + c_5^2p_2^2$	(p_1, p_3^2)
	16	$5p_2^5$	(p_1, p_3, p_4, c_5)
	18	$5c_5p_4^2 + 9c_5p_4p_2^2 + 7c_5p_2^4$	(p_1, p_3, c_5^2)
	24	$p_4^3 + 4p_4^2p_2^2 + 12p_4p_2^4 + 7p_2^6$	(p_1, p_3, c_5)
17	4	$2p_4p_3p_2 + 16p_3p_2^3 + 16c_5^2p_4 + c_5^2p_2^2$	(p_1, p_3^2)
	10	$4c_5p_4^2 + 9c_5p_4p_2^2 + 2c_5p_2^4$	(p_1, p_3, c_5^2)
	16	$11p_4^3 + p_4^2p_2^2 + 8p_4p_2^4 + 8p_2^6$	(p_1, p_3, c_5)

We finally calculate $\mathcal{P}^1x_k$ for a generator $x_k \in H^*(B\mathrm{G}_2;\mathbb{Z}/p)$.

Proposition 2.15. *For a generator $x_k \in H^*(B\mathrm{G}_2;\mathbb{Z}/p)$, we have*

$$\mathcal{P}^1x_k = \begin{cases} x_4x_{12} + 2x_4^4 & (k,p) = (4,7) \\ 6x_{12}^2 + 2x_4^3x_{12} & (k,p) = (12,7) \\ 6x_{12}^2 + x_4^3x_{12} + 2x_4^6 & (k,p) = (4,11). \end{cases}$$

Proof. By Proposition 2.10 and the naturality of $\mathcal{P}^1$, we have $\mathcal{P}^1x_{4k} = \mathcal{P}^1\rho^*(p_k) = \rho^*(\mathcal{P}^1p_k)$, hence the proof is completed by Lemma 2.11. □

3. Proof of Theorem 2.2

In this section, we prove Theorem 2.2 by using results in the previous section.

3.1. The case of E_8

Suppose that E_8 is p-regular, that is, $p > 30$. By an easy degree consideration, we see that if $\mathcal{P}^1x_k \mod (x_{2i}\,|\,i \in \mathtt{t}(\mathrm{E}_8))^3$ is nontrivial for a generator x_k of $H^*(B\mathrm{E}_8;\mathbb{Z}/p)$, it is as in the following table.

	$\mathcal{P}^1 x_k \mod (x_{2i}\,|\,i\in\mathtt{t}(\mathrm{E}_8))^3$	(k,p)
(1)	$\lambda_1x_4x_{60}+\lambda_2x_{16}x_{48}+\lambda_3x_{24}x_{40}+\lambda_4x_{28}x_{36}$	$(4,31)$
(2)	$\lambda_1x_{16}x_{60}+\lambda_2x_{28}x_{48}+\lambda_3x_{36}x_{40}$	$(16,31),(4,37)$
(3)	$\lambda_1x_{24}x_{60}+\lambda_2x_{36}x_{48}$	$(24,31),(4,41)$
(4)	$\lambda_1x_{28}x_{60}+\lambda_2x_{40}x_{48}$	$(28,31),(16,37),(4,43)$
(5)	$\lambda_1x_{36}x_{60}+\lambda_2x_{48}^2$	$(36,31),(24,37),(16,41),(4,47)$
(6)	$\lambda x_{40}x_{60}$	$(40,31),(28,37),(16,43)$
(7)	$\lambda x_{48}x_{60}$	$(48,31),(36,37),(28,41),(24,43),$ $(16,47),(4,53)$
(8)	λx_{60}^2	$(60,31),(48,37),(40,41),(36,43),$ $(28,47),(16,53),(4,59)$

Let I_k for $k=1,\dots,8$ be the ideals of $\mathbb{Z}/p[p_1,\dots,p_7,c_8]$ as in Proposition 2.12.
(1) It is proved in [HK] that $\lambda_i\ne 0$ for $i=1,2,3,4$.
(2) Since $\hat{x}_i\in I_1$ for $i=4,16,24$, for a degree reason, we have

$$\rho_1^*(\mathcal{P}^1x_k)\equiv\lambda_2\hat{x}_{28}\hat{x}_{48}+\lambda_3\hat{x}_{36}\hat{x}_{40}\equiv-4000(24\lambda_2p_7^2p_5+(\lambda_2-6\lambda_3)p_7p_5^2p_2)\mod I_1+(p_4).$$

On the other hand, by the naturality of $\mathcal{P}^1$ and Proposition 2.12,

$$\rho_1^*(\mathcal{P}^1x_k)=\mathcal{P}^1\rho_1(x_k)\equiv\begin{cases}24p_7p_5^2p_2+9p_7^2p_5 & (p=31)\\ 34p_7p_5^2p_2+p_7^2p_5 & (p=37)\end{cases}\mod I_1+(p_4),$$

implying that $(\lambda_2,\lambda_3)=(19,2),(5,30)$ according as $p=31,37$. Since $\hat{x}_4,\hat{x}_{16}^2,\hat{x}_{24},\hat{x}_{36}\in I_1+(p_2,p_7)$, we also have

$$\rho_1^*(\mathcal{P}^1x_k)\equiv\lambda_1\hat{x}_{16}\rho_1^*(x_{60})+\lambda_3\hat{x}_{36}\hat{x}_{40}\equiv\lambda_1\hat{x}_{16}\rho_1^*(x_{60})-1500\lambda_3p_5^3p_4\mod I_1+(p_2,p_7),$$

and by Proposition 2.12,

$$\rho_1^*(\mathcal{P}^1x_k)=\mathcal{P}^1\rho_1^*(x_k)\equiv\begin{cases}22p_5^3p_4 & (p=31)\\ 36p_5^3p_4 & (p=37)\end{cases}\mod I_1+(p_2,p_7).$$

Then we see that $\lambda_1\hat{x}_{16}\rho_1^*(x_{60})\equiv(1500\lambda_3+\delta)p_5^3p_4\not\equiv 0\mod I_1+(p_2,p_7)$ for $\delta=22,36$ according as $p=31,37$, implying $\lambda_1\ne 0$.

(3) Since $\hat{x}_i, \hat{x}_j^2 \in I_2$ for $i = 4, 16$ and $j = 24, 28, 36$, we have

$$\rho_1^*(\mathcal{P}^1 x_k) \equiv \lambda_1 \hat{x}_{24} \rho_1^*(x_{60}) + \lambda_2 \hat{x}_{36} \hat{x}_{48} \equiv \lambda_1 \hat{x}_{24} \rho_1^*(x_{60}) - 14400 \lambda_2 p_7 p_6 p_5 p_3 \mod I_2.$$

By the naturality of $\mathcal{P}^1$ and Proposition 2.12, we also have

$$\rho_1^*(\mathcal{P}^1 x_k) = \mathcal{P}^1 \rho_1^*(x_k) \equiv \begin{cases} 28 p_7 p_6 p_5 p_3 + 16 p_6 p_5^3 & (p = 31) \\ 35 p_7 p_6 p_5 p_3 + 40 p_6 p_5^3 & (p = 41) \end{cases} \mod I_2,$$

implying that $\lambda_1 \neq 0$ and $\lambda_2 \neq 0$ for both $p = 31, 41$.
(4) Since $\hat{x}_i, \hat{x}_{28}^2 \in I_3 + (p_3, p_4, p_7^2, \hat{x}_{40})$ for $i = 4, 16, 24, 36, 40$, it follows from Proposition 2.12 that

$$\lambda_1 \hat{x}_{28} \rho_1^*(x_{60}) \equiv \rho_1^*(\mathcal{P}^1 x_k) \equiv \mathcal{P}^1 \rho_1^*(x_k) \not\equiv 0 \mod I_3 + (p_3, p_4, p_7^2, \hat{x}_{40})$$

so $\lambda_1 \neq 0$. We can similarly get $\lambda_2 \neq 0$ by considering $\rho_1^*(\mathcal{P}^1 x_k) \mod I_3 + (p_7^2, \hat{x}_{28})$ since $\hat{x}_i \in I_3 + (p_7^2, \hat{x}_{28})$ for $i = 4, 16, 24, 28$.
(5), (6) and (7) We get $\lambda \neq 0$ similarly to (4) by considering $\rho_1^*(\mathcal{P}^1 x_k)$ modulo the ideals $I_4 + (p_7), I_5, I_6 + (\hat{x}_{40}^2)$ respectively for (5), (6) and (7) since $\hat{x}_4, \hat{x}_{16}, \hat{x}_{24}^2, \hat{x}_{36}^2 \in I_4 + (p_7)$, $\hat{x}_i \in I_5$ for $i = 4, 16, 24, 18, 36$ and $\hat{x}_i \in I_6 + (\hat{x}_{40}^2)$ for $i = 4, 16, 24, 36, 40$.
(8) Suppose $(k, p) \neq (60, 31)$. Since $\hat{x}_i, \hat{x}_{28}^2, \hat{x}_{40}^3 \in I_7 + (\hat{x}_{40}^3)$ for $i = 4, 16, 24, 36$, we get $\lambda \neq 0$ by considering $\rho_1^*(\mathcal{P}^1 x_k) \mod I_7 + (\hat{x}_{40}^3)$ as above.

Suppose next that $(k, p) = (60, 31)$. By a degree reason, we have

$$\rho_1^*(x_{60}) \equiv \alpha p_5^3 + \beta p_7 p_5 p_3 \mod I_8 + (\hat{x}_{40}^2)$$

for $\alpha, \beta \in \mathbb{Z}/p$. Since $\hat{x}_i, \hat{x}_{40}^2 \in I_8 + (\hat{x}_{40}^2)$ for $i = 4, 16, 24, 36$ and $\rho_1^*(x_{48}) \equiv -200 p_7 p_5 \mod I_8$, we have

$$\rho_1^*(\mathcal{P}^1 x_{48}) \equiv \mu \hat{x}_{48} \rho_1^*(x_{60}) \equiv -200 \mu (\alpha p_7 p_5^4 + \beta p_7^2 p_5^2 p_3) \mod I_8 + (\hat{x}_{40}^2)$$

for some $\mu \in \mathbb{Z}/p$. By Proposition 2.12, we also have

$$\rho_1^*(\mathcal{P}^1 x_{48}) = \mathcal{P}^1 \rho_1^*(x_{48}) \equiv 10 p_7 p_5^4 + 11 p_7^2 p_5^2 p_3 \mod I_8 + (\hat{x}_{40}^2)$$

Then we may put $(\alpha, \beta) = (17, 28)$ and $\mu = 1$. In the case (7), we have seen that $\mathcal{P}^1 x_{48} \equiv \mu x_{48} x_{60} \mod (x_{2i} \,|\, i \in \mathtt{t}(\mathrm{E}_8))^3$, implying that $\mathcal{P}^1 \mathcal{P}^1 x_{48} \equiv (\lambda + 1) x_{48} x_{60}^2 \mod (x_{2i} \,|\, i \in \mathtt{t}(\mathrm{E}_8))^4$, where $\mathcal{P}^1 x_{60} \equiv \lambda x_{60}^2 \mod (x_{2i} \,|\, i \in \mathtt{t}(\mathrm{E}_8))^3$. Then for a degree reason, we get

$$\rho_1^*(\mathcal{P}^1 \mathcal{P}^1 x_{48}) \equiv (\lambda + 1) \hat{x}_{48} \rho_1^*(x_{60})^2 \equiv 21 (\lambda + 1) p_7^3 p_5^3 p_3^2 \mod I_8 + (\hat{x}_{40}^2).$$

On the other hand, by the Adem relation $\mathcal{P}^1\mathcal{P}^1 = 2\mathcal{P}^2$ and Proposition 2.12, we have

$$\rho_1^*(\mathcal{P}^1\mathcal{P}^1 x_{48}) = \rho_1^*(2\mathcal{P}^2 x_{48}) = 2\mathcal{P}^2\rho_1^*(x_{48}) \equiv 7p_7^3p_5^3p_3^2 \mod I_8 + (\hat{x}_{40}^2),$$

hence $\lambda \neq 0$.

3.2. The case of E_7

Suppose that E_7 is p-regular, that is, $p > 18$. Then if $\mathcal{P}^1 x_k \mod (x_{2i} \,|\, i \in \mathtt{t}(\mathrm{E}_7))^3$ is non-trivial, it is as in the following table.

	$\mathcal{P}^1 x_k \mod (x_{2i} \,\vert\, i \in \mathtt{t}(\mathrm{E}_7))^3$	(k, p)
(1)	$\lambda_1 x_4 x_{36} + \lambda_2 x_{12} x_{28} + \lambda_3 x_{16} x_{24} + \lambda_4 x_{20}^2$	$(4, 19)$
(2)	$\lambda_1 x_{12} x_{36} + \lambda_2 x_{20} x_{28} + \lambda_3 x_{24}^2$	$(12, 19), (4, 23)$
(3)	$\lambda_1 x_{16} x_{36} + \lambda_2 x_{24} x_{28}$	$(16, 19)$
(4)	$\lambda_1 x_{20} x_{36} + \lambda_2 x_{28}^2$	$(20, 19), (12, 23)$
(5)	$\lambda x_{24} x_{36}$	$(24, 19), (16, 23), (4, 29)$
(6)	$\lambda x_{28} x_{36}$	$(28, 19), (4, 31)$
(7)	λx_{36}^2	$(36, 19), (28, 23), (16, 29), (12, 31)$

(1) It is proved in [HK] that $\lambda_i \neq 0$ for $i = 1, 2, 3, 4$.
(2) Put $I = (p_1, p_3^2, c_6, \hat{x}_{16})$. Since $\hat{x}_4, \hat{x}_{12}^2, \hat{x}_{16} \in I$, by Corollary 2.7, we have

$$\rho_2^*(\mathcal{P}^1 x_k) \equiv \lambda_1\hat{x}_{12}\hat{x}_{36} + \lambda_2\hat{x}_{20}\hat{x}_{28} + \lambda_3\hat{x}_{24}^2 \equiv 60\lambda_1 p_5p_3p_2^2 + 40\lambda_2 p_5^2p_2 + \tfrac{25}{81}\lambda_3 p_2^6 \mod I.$$

On the other hand, it follows from Proposition 2.13 that

$$\rho_2^*(\mathcal{P}^1 x_k) = \mathcal{P}^1\rho_2(x_k) \equiv \begin{cases} 18p_5^2p_2 + 10p_5p_3p_2^2 + p_2^6 & (p = 19) \\ 22p_5^2p_2 + 7p_5p_3p_2^2 + 7p_2^6 & (p = 23) \end{cases} \mod I,$$

hence $\lambda_1 \neq 0$, $\lambda_2 \neq 0$ and $\lambda_3 \neq 0$.
(3) In this case, we have $(k, p) = (16, 19)$. Put $I = (p_1, p_3, c_6, \hat{x}_{16}^2)$. Since $\hat{x}_4, \hat{x}_{12}, \hat{x}_{16}^2 \in I$, it follows from Proposition 2.7 that

$$\rho_2^*(\mathcal{P}^1 x_{16}) \equiv \lambda_1\hat{x}_{16}\hat{x}_{36} + \lambda_2\hat{x}_{24}\hat{x}_{28} \equiv (13\lambda_1 + 9\lambda_2)p_5p_4p_2^2 + (9\lambda_1 + 14\lambda_2)p_5p_2^4 \mod I.$$

By Proposition 2.13, we also have $\rho_2^*(\mathcal{P}^1 x_{16}) = \mathcal{P}^1\rho_2^*(x_{16}) \equiv 11p_5p_4p_2^2 + 14p_5p_2^4 \mod I$, implying $\lambda_1 \neq 0$ and $\lambda_2 \neq 0$.

(4) Put $I=(p_1,p_3^2,c_6^2,\hat{x}_{12},\hat{x}_{16}^2,\hat{x}_{24},\hat{x}_{16}\hat{x}_{20}^2)$. Since $\hat{x}_i,\hat{x}_{16}^2,\hat{x}_{16}\hat{x}_{20}^2\hat{x}_{24}\in I$ for $i=4,12,24$, we have

$$\rho_2^*(\mathcal{P}^1x_k)\equiv\lambda_1\hat{x}_{20}\hat{x}_{36}+\lambda_2^2\hat{x}_{28}^2\equiv(-10\lambda_1+1600\lambda_2)p_5^2p_2^2+(\tfrac{2}{3}\lambda_1-\tfrac{320}{3}\lambda_2)p_5p_3p_2^3 \mod I.$$

By Proposition 2.13, we also have

$$\rho_2^*(\mathcal{P}^1x_k)=\mathcal{P}^1\rho_2^*(x_k)\equiv\begin{cases}10p_5^2p_2^2+12p_5p_3p_2^3 & (p=19)\\ 15p_5^2p_2^2+22p_5p_3p_2^3 & (p=23)\end{cases} \mod I,$$

hence $\lambda_1\neq 0$ and $\lambda_2\neq 0$.
(5) and (7) Put $I=(p_1,p_3,p_5^2,c_6,\hat{x}_{16})$ and $J=(p_1,p_3^2,c_6^2,\hat{x}_{12},\hat{x}_{16},\hat{x}_{20}^3,\hat{x}_{24}^2,\hat{x}_{20}\hat{x}_{24}\hat{x}_{28})$. Then since $\hat{x}_i,\hat{x}_{20}^2\in I$ for $i=4,12,16$ and $\hat{x}_i,\hat{x}_{20}^3,\hat{x}_{24}^2,\hat{x}_{20}\hat{x}_{24}\hat{x}_{28}\in J$ for $i=4,12,16$, we have $\lambda\neq 0$ similarly to (4) of E_8 by considering $\rho_2^*(\mathcal{P}^1x_k)$ modulo I and J respectively for (5) and (7).
(6) The case $p=31$ follows from the above case of E_8 together with Corollary 2.7. Then we consider the case $p=19$. Put $I=(p_1,p_3^2,c_6^2,\hat{x}_{12},\hat{x}_{16},\hat{x}_{20}^2,\hat{x}_{24}^2)$. Since $\hat{x}_i,\hat{x}_j^2\in I$ for $i=4,12,16$ and $j=20,24$, we get $\lambda\neq 0$ as above by considering $\rho_2^*(\mathcal{P}^1x_k) \mod I$.

3.3. The cases of E_6 *and* F_4

We first consider the case of E_6. Suppose that E_6 is p-regular, that is, $p\geq 13$. By an easy dimensional consideration, we see that if $\mathcal{P}^1x_k\not\equiv 0 \mod (x_{2i}\,|\,i\in\mathtt{t}(\mathrm{E}_6))^3$, it is as in the following table.

	$\mathcal{P}^1x_k \mod (x_{2i}\,\vert\,i\in\mathtt{t}(\mathrm{E}_6))^3$	(k,p)
(1)	$\lambda_1x_4x_{24}+\lambda_2x_{10}x_{18}+\lambda_3x_{12}x_{16}$	$(4,13)$
(2)	$\lambda_1x_{10}x_{24}+\lambda_2x_{16}x_{18}$	$(10,13)$
(3)	$\lambda_1x_{12}x_{24}+\lambda_2x_{18}^2$	$(12,13),(4,17)$
(4)	$\lambda x_{16}x_{24}$	$(16,13),(4,19)$
(5)	$\lambda x_{18}x_{24}$	$(18,13),(10,17)$
(6)	λx_{24}^2	$(24,13),(16,17),(12,19),(4,23)$

When $p=19,23$, the result follows from the above case of E_7 and Corollary 2.8.
(1) It is proved in [HK] that $\lambda_1\neq 0$, $\lambda_2\neq 0$ and $\lambda_3\neq 0$.

(2) Put $I=(p_1,p_3^2,c_5^2)$. Since $\hat{x}_4,\hat{x}_{10}^2,\hat{x}_{12}^2\in I$, we have

$$\rho_3^*(\mathcal{P}^1x_{10})\equiv\lambda_1\hat{x}_{10}\hat{x}_{24}+\lambda_2\hat{x}_{16}\hat{x}_{18}\equiv 5\lambda_1(-p_4p_2c_5+\tfrac{1}{36}p_2^3c_5)+\lambda_2(12p_4p_2c_5+p_2^3c_5)\mod I,$$

where $\hat{x}_{10}=c_5$ and $\hat{x}_{18}=p_2c_5$. On the other hand, by Proposition 2.14, we have $\rho_3^*(\mathcal{P}^1x_{10})=\mathcal{P}^1\rho_3^*(x_{10})\equiv 6p_4p_2c_5+7p_2^3c_5\mod I$ for $p=13$, hence $\lambda_1\neq 0$ and $\lambda_2\neq 0$.
(3) Put $I=(p_1,p_3^2,\hat{x}_{16})$. It is sufficient to consider the case $p=13,17$. Since $\hat{x}_i,\hat{x}_{12}^2\in I$ for $i=4,16$,

$$\rho_3^*(\mathcal{P}^1x_k)\equiv\lambda_1\hat{x}_{12}\hat{x}_{24}+\lambda_2\hat{x}_{18}^2\equiv-\tfrac{10}{3}\lambda_1p_3p_2^3+\lambda_2p_2^2c_5^2\mod I.$$

By Proposition 2.14, we have

$$\rho_3^*(\mathcal{P}^1x_k)=\mathcal{P}^1\rho_3^*(x_k)\equiv\begin{cases}9p_3p_2^3+5c_5^2p_2^2 & (p=13)\\ 13p_3p_2^3-11c_5^2p_2^2 & (p=17)\end{cases}\mod I,$$

implying $\lambda_1\neq 0$ and $\lambda_2\neq 0$.
(4), (5) and (6) Put $I=(p_1,p_3,p_4,c_5)$, $J=(p_1,p_3,c_5^2,\hat{x}_{16})$ and $K=(p_1,p_3,c_5,\hat{x}_{16})$. Then since $\hat{x}_i\in I$ for $i=4,10,12$, $\hat{x}_i,\hat{x}_{10}^2\in J$ for $i=4,12,16$ and $\hat{x}_i\in K$ for $i=4,12,10,16$, we get $\lambda\neq 0$ similarly to (4) of E_8 by considering $\rho_3^*(\mathcal{P}^1x_k)$ modulo I,J,K respectively for (4), (5) and (6).

We next consider the case of F_4. Notice that F_4 is p-regular if and only if so is E_6, and that as in the proof of Corollary 2.9, the map $\alpha_3^*: H^*(B\mathrm{E}_6;\mathbb{Z}/p)\to H^*(B\mathrm{F}_4;\mathbb{Z}/p)$ is surjective. Then the result for F_4 follows from that for E_6 above.

3.4. *The case of* G_2

For a degree reason, if G_2 is p-regular and $\mathcal{P}^1x_k\not\equiv 0\mod (x_{2i}\,|\,i\in\mathtt{t}(\mathrm{G}_2))^3$, then $(k,p)=(4,7),(12,7),(4,11)$. Hence Theorem 2.2 for G_2 readily follows from Proposition 2.15.

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